ECGA 6310
Practice Problems

Problem 1. Life Cycle Consumption with Quadratic Utility
Suppose that the consumer maximizes the following objective function:

$$\max \sum_{t=0}^{T} \beta^t [u(c_t)] \tag{1}$$

subject to the dynamic budget constraint (Note that this is written slightly different than we did in class; namely income is assumed to be received at the end of period t rather than at the beginning of period $t+1$):

$$A_{t+1} = (1+r)[A_t + y_t - c_t] \tag{2}$$

where

$$u(c_t) = c_t - \frac{b}{2} c_t^2 \tag{3}$$

with $b > 0$. r is a constant net interest rate, A_t is the amount of wealth available at the beginning of period t and y_t is labour income in period t.

1. Do we need a transversality condition? If not, what other constraint do we need? How much wealth should the consumer have when she dies?

2. Assume the agent receives income A_0 in the first period. Show that the dynamic budget constraint together with the constraint you found in (1) implies the following intertemporal budget constraint:

$$\sum_{t=0}^{T} R^{-t} c_t = A_0 + \sum_{t=0}^{T} R^{-t} Y_t \tag{4}$$

where $R = (1+r)$. Note: Begin with the dynamic budget constraint (dbc) describing A_1. Then substitute this expression for A_1 into the dbc for A_2. Next substitute the expression you got for A_2 into the dbc for period 3, etc. This will allow you to derive the intertemporal budget constraint.

3. Set up the consumer's maximization problem. What is Bellman's Equation?

4. Find the first order conditions (use the Envelope theorem).

5. Derive the Euler Equation.

6. Derive a closed form solution for c_0.

Problem 2. Life Cycle Utility over Consumption and Leisure Suppose that the consumer maximizes the following objective function:

$$\max \sum_{t=0}^{T} \beta^t [u(c_t) + d(l_t)] \tag{5}$$

where c_t is consumption and l is leisure. Assume that both functions, u and d are increasing and concave.

Consumption is typically measured using expenditure x_t on some group of goods and services. In reality, consumption requires expenditure plus time spent on 'home production'and households can substitute between the two. For example, it is more expensive to eat out or buy pre-prepared food, but it is less costly in terms of time. It is cheaper to buy raw foods and prepare them yourself but it is more costly in terms of time.

Assume that consumption is related to expenditure and time inputs:

$$c_t = f(x_t, s_t) \tag{6}$$

where s_t represents time spent in home production. Assume that f is increasing and concave in both of its inputs. Assume that the household can borrow or lend at the risk free rate, $R = 1 + r$. The household's dynamic budget constraint is

$$A_{t+1} = R[A_t + w_t h_t - x_t] \tag{7}$$

where h_t is hours spend working outside the home, w_t is the wage rate, which is exogenous, A_t is financial assets at the beginning of period t, and x_t is expenditure. There is no uncertainty, at time 0 households know the entire sequence of wages with certainty from 0 to T.

Assume that households have an endowment of one unit of time each period, which they spend on leisure, working for wages, and in home production. Thus leisure time satisfies:

$$l_t = 1 - h_t - s_t \tag{8}$$

1. What are the state variables? What are the control variables?
2. Write down the Bellman equation for this problem.
3. Derive the first order conditions (hint: there will be 3 in addition to the envelope condition)
4. Find the Euler Equation.
5. Show that $\frac{x_t}{s_t}$ is an increasing function of wages, w_t. That is, as wages rise people substitute expenditures for time in the production of consumption. Provide intuition for this result.

Problem 3. Assume a quadratic utility, rational expectations framework and assume that the rate of time preference, ρ equals the interest rate, r. Assume that labour income follows the following stochastic process:

$$y_{t+1} = \lambda y_t + (1 - \lambda)\bar{y} + \epsilon_{t+1} \tag{9}$$

where $E_t \epsilon_{t+1} = 0$ and ϵ_{t+1} is an income innovation, $0 \geq \lambda \leq 1$and $\bar{y}$ is the unconditional mean of labour income.

1. Prove that the consumption function in this case has the following form:

$$c_t = rA_t + \frac{r}{1 + r - \lambda} y_t + \frac{1 - \lambda}{1 + r - \lambda} \bar{y}. \tag{10}$$

2. What happens if $\lambda = 1$? Explain.
3. What happens if $\lambda = 0$? Explain.